Propositional Logic

Question: How do we formalize the definitions and reasoning we use in our proofs?

Where We're Going

- ***Propositional Logic*** (Today)
  - Reasoning about Boolean values.
- ***First-Order Logic*** (Wednesday/Friday)
  - Reasoning about properties of multiple objects.

Propositional Logic

A ***proposition*** is a statement that is either true or false.

In other words, English sentences can be propositions, but not all are (for example, commands and questions can't be propositions).

Propositional Logic

- ***Propositional logic*** is a mathematical system for reasoning about propositions and how they relate to one another.
- Every statement in propositional logic consists of ***propositional variables*** combined via ***propositional connectives***.
  - Each variable represents some proposition, such as “The integer x is odd.” or “You got an A+ in CS103.”
  - Connectives encode how propositions are related, such as “If x is odd, then x^2 is odd.”

Algebra: it generalizes reasoning about arithmetic

- In elementary school arithmetic, we learn that two expressions are equivalent, *for specific numbers*:

$$(9 + 5) / 7 = 14 / 7 = 2$$

$$(1/7)(9 + 5) = (1/7)(14) = 2$$

- In high school, you leveled up to algebra, which lets us study the **guaranteed-reusable structural patterns** of equivalence, which always apply *regardless of the specific numbers involved*. So we can see that the above math generalizes to a reusable rule:

$$(a + b) / c = (1/c)(a + b)$$

- Algebra replaces the numbers with variables so we can focus on analyzing and manipulating the structure.

Propositional Logic as a Boolean Algebra

- Philosophers, mathematicians, and logicians wanted to do the same kind of numbers → algebra level-up that arithmetic enjoys, but in the space of *arguments*.
- Just like algebra replaces specific numbers with variables, propositional logic replaces specific English sentences with propositional variables.
- So we can focus on analyzing and manipulating the structure to create and apply generalizable rules about which argument structures are valid.
  - *Example generalizable argument rule:* if you prove the contrapositive of a statement is true, you've proved the original statement is true.

Propositional Variables

- Each proposition (sentence) will be represented by a ***propositional variable***.
- Propositional variables are usually represented as lower-case letters, such as *p, q, r, s,* etc.
- Each variable can take one one of two values: true or false.

Propositional Connectives

- There are seven propositional connectives, many of which will be familiar from programming.
- First, there's the logical "NOT" operation:

$$\neg p$$

- You'd read this out loud as "not p."
- The fancy name for this operation is ***logical negation***.

Propositional Connectives

- There are seven propositional connectives, many of which will be familiar from programming.
- Next, there's the logical "AND" operation:

$$p \wedge q$$

- You'd read this out loud as "p and q."
- The fancy name for this operation is ***logical conjunction***.

Propositional Connectives

- There are seven propositional connectives, many of which will be familiar from programming.
- Then, there's the logical "OR" operation:

$$p \vee q$$

- You'd read this out loud as "*p* or *q*."
- The fancy name for this operation is ***logical disjunction***. This is an *inclusive* or.

Truth Tables

- A ***truth table*** is a table showing the truth value of a propositional logic formula as a function of its inputs.
- Let's go look at the truth tables for the three connectives we've seen so far:

¬ ∧ ∨

Quick check: how many rows of the truth table output are true for ∨?
Go to PollEv.com/cs103spr26

Summary of Important Points

- The ∨ connective is an *inclusive* "or." It's true if at least one of the two operands is true.
  - Similar to the **||** operator in C, C++, Java, etc. and the **or** operator in Python.
- If we need an *exclusive* "or" operator, we can build it out of the connectives we already have.

Summary of Important Points

- The ∨ connective is an *inclusive* "or." It's true if at least one of the two operands is true.
  - Similar to the **||** operator in C, C++, Java, etc. and the **or** operator in Python.
- If we need an *exclusive* "or" operator, we can build it out of the connectives we already have.

Quick check: Try this yourself! Take a minute to combine the operators **¬ ∧ ∨** together to form an expression that represents p **exclusive-or** q (something that's true if and only if exactly one of p and q is true—hint the truth table should have 2 true rows)
Go to PollEv.com/cs103spr26

Mathematical Implication

Implication

- We can represent implications using this connective:

$$\boldsymbol{p \rightarrow q}$$

- You'd read this out loud as "p implies q" or "if p then q."
- ***Question:*** What should the truth table for $p \rightarrow q$ look like?

Implication

Dr. Bailey: "**If** you pick a perfect March Madness bracket this year, **then** I'll give you an A+ in CS103."

What if…

- …you pick a bad bracket and get a C?
- …you pick a bad bracket and get an A+?
- …you pick a perfect bracket and get a C?
- …you pick a perfect bracket and get an A+?

Implication

- ...you pick a bad bracket and get a C?
- ...you pick a bad bracket and get an A+?
- ...you pick a perfect bracket and get a C?
- ...you pick a perfect bracket and get an A+?

p	q	$p \rightarrow q$
F	F	
F	T	
T	F	
T	T	

Implication

- ...you pick a bad bracket and get a C?
- ...you pick a bad bracket and get an A+?
- ...you pick a perfect bracket and get a C?
- ...you pick a perfect bracket and get an A+?

p	q	$p \rightarrow q$
F	F	
F	T	
T	F	
T	T	

Implication

- ...you pick a bad bracket and get a C?
- ...you pick a bad bracket and get an A+?
- ...you pick a perfect bracket and get a C?
- ...you pick a perfect bracket and get an A+?

p	q	$p \rightarrow q$
F	F	T
F	T	
T	F	
T	T	

Implication

- ...you pick a bad bracket and get a C?
- ...you pick a bad bracket and get an A+?
- ...you pick a perfect bracket and get a C?
- ...you pick a perfect bracket and get an A+?

p	q	$p \rightarrow q$
F	F	T
F	T	
T	F	
T	T	

Implication

- ...you pick a bad bracket and get a C?
- ...you pick a bad bracket and get an A+?
- ...you pick a perfect bracket and get a C?
- ...you pick a perfect bracket and get an A+?

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	
T	T	

Implication

- ...you pick a bad bracket and get a C?
- ...you pick a bad bracket and get an A+?
- ...you pick a perfect bracket and get a C?
- ...you pick a perfect bracket and get an A+?

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	
T	T	

Implication

- ...you pick a bad bracket and get a C?
- ...you pick a bad bracket and get an A+?
- ...you pick a perfect bracket and get a C?
- ...you pick a perfect bracket and get an A+?

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	

Implication

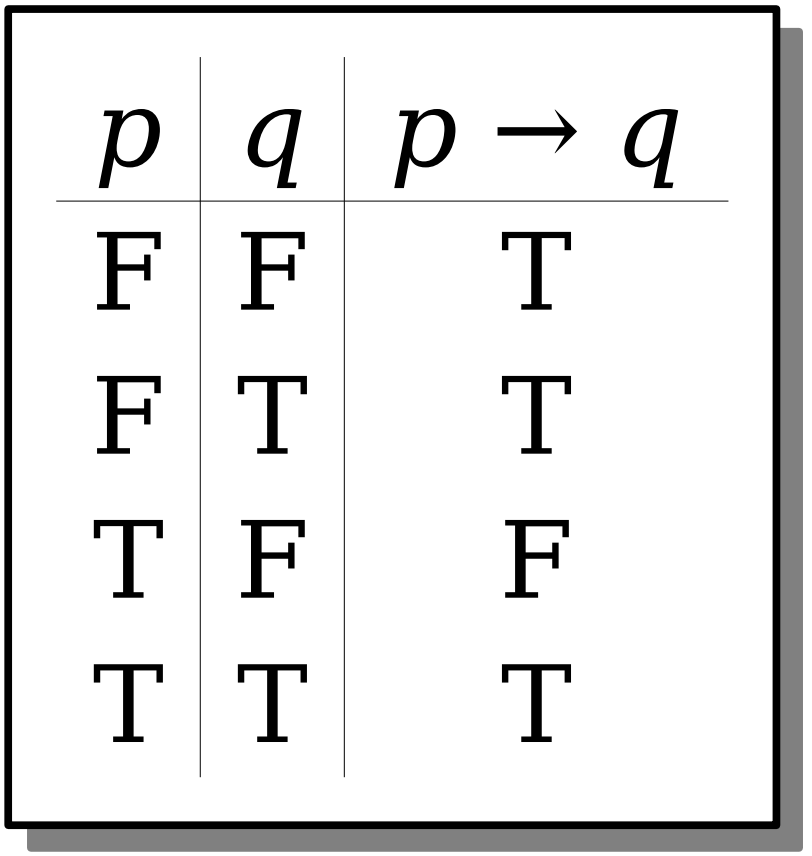
- ...you pick a bad bracket and get a C?
- ...you pick a bad bracket and get an A+?
- ...you pick a perfect bracket and get a C?
- ...you pick a perfect bracket and get an A+?

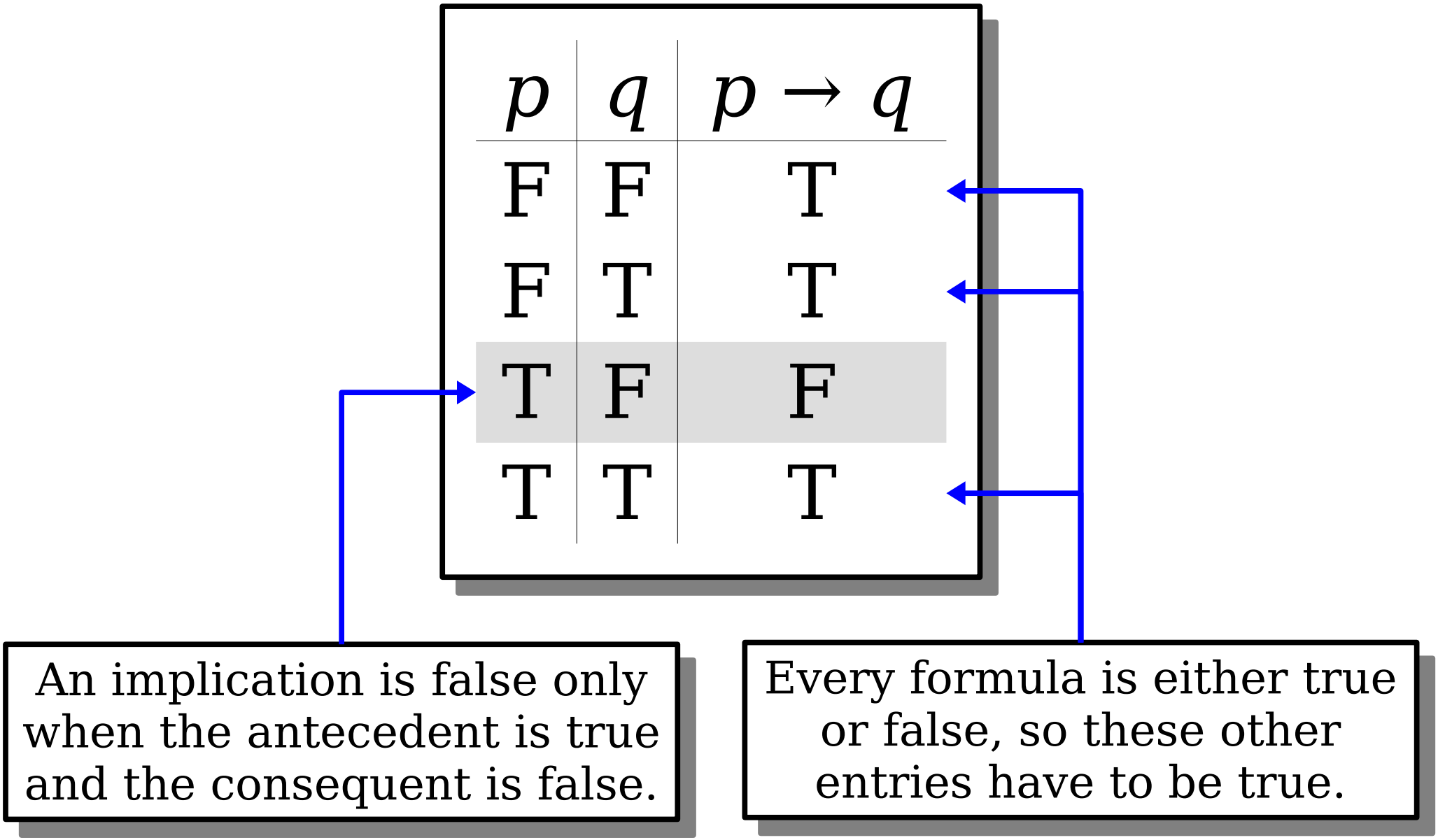
p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	

Implication

- ...you pick a bad bracket and get a C?
- ...you pick a bad bracket and get an A+?
- ...you pick a perfect bracket and get a C?
- ...you pick a perfect bracket and get an A+?

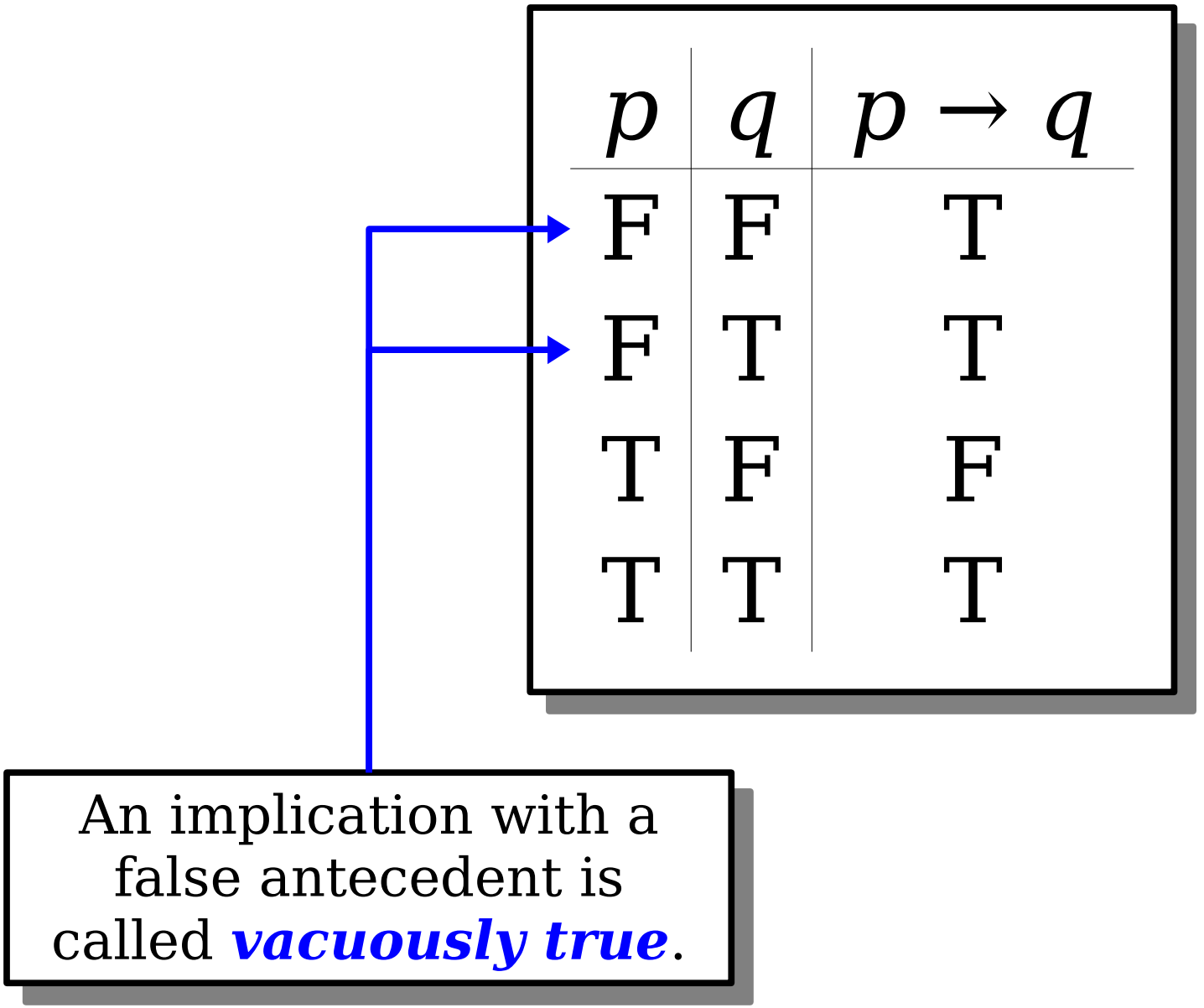
p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T


p q p → q
F F T
F T T
T F F
T T T


p q $p \to q$
F F T
F T T
T F F
T T T
An implication is false only when the antecedent is true and the consequent is false.
Every formula is either true or false, so these other entries have to be true.

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

Important observation:
The statement $p \to q$ is true whenever $p \wedge \neg q$ is false.


p q $p \rightarrow q$
F F T
F T T
T F F
T T T
An implication with a false antecedent is called ***vacuously true***.

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

Please commit this table to memory. We're going to need it, extensively, over the next couple of weeks.

The Biconditional Connective

The Biconditional Connective

- On Friday, we saw that "p if and only if q" means both that $p \rightarrow q$ and $q \rightarrow p$.
- We can write this in propositional logic using the ***biconditional*** connective:

$$\boldsymbol{p \leftrightarrow q}$$

- This connective's truth table has the same meaning as "p implies q and q implies p."
- Based on that, what should its truth table look like?
- Take a guess, and talk it over with your neighbor!

Biconditionals

- The ***biconditional*** connective $p \leftrightarrow q$ is read "p if and only if q."
- Here's its truth table:

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Biconditionals

- The ***biconditional*** connective $p \leftrightarrow q$ is read "p if and only if q."
- Here's its truth table:

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

One interpretation of ↔ is to think of it as equality: the two propositions must have equal truth values.

True and False

- There are two more logic symbols to learn: true and false.
  - The symbol $\top$ is a value that is always true.
  - The symbol $\bot$ is value that is always false.

Fun Fact: Logic of the Proof by Contradiction

- Suppose you want to prove p is true using a proof by contradiction.
- The setup looks like this:
  - Assume p is false.
  - Derive something that we know is false.
  - Conclude that p is true.
- In propositional logic:

$$\mathbf{(\neg p \to \bot) \to p}$$

Operator Precedence

- How do we parse this statement?

$$\neg x \to y \vee z \to x \vee y \wedge z$$

- Operator precedence for propositional logic:

$\neg$

$\wedge$

$\vee$

$\to$

$\leftrightarrow$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$\neg x \to y \vee z \to x \vee y \wedge z$$

- Operator precedence for propositional logic:

$\neg$

$\wedge$

$\vee$

$\to$

$\leftrightarrow$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to y \vee z \to x \vee y \wedge z$$

- Operator precedence for propositional logic:

$\neg$

$\wedge$

$\vee$

$\to$

$\leftrightarrow$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to y \vee z \to x \vee y \wedge z$$

- Operator precedence for propositional logic:

¬

∧

∨

→

↔

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow y \vee z \rightarrow x \vee (y \wedge z)$$

- Operator precedence for propositional logic:

$\neg$

$\wedge$

$\vee$

$\rightarrow$

$\leftrightarrow$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow y \vee z \rightarrow x \vee (y \wedge z)$$

- Operator precedence for propositional logic:

$\neg$

$\wedge$

$\vee$

$\rightarrow$

$\leftrightarrow$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to (y \vee z) \to (x \vee (y \wedge z))$$

- Operator precedence for propositional logic:

$\neg$

$\wedge$

$\vee$

$\to$

$\leftrightarrow$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow (y \vee z) \rightarrow (x \vee (y \wedge z))$$

- Operator precedence for propositional logic:

$\neg$

$\wedge$

$\vee$

$\rightarrow$

$\leftrightarrow$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow ((y \vee z) \rightarrow (x \vee (y \wedge z)))$$

- Operator precedence for propositional logic:

$\neg$

$\wedge$

$\vee$

$\rightarrow$

$\leftrightarrow$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow ((y \vee z) \rightarrow (x \vee (y \wedge z)))$$

- Operator precedence for propositional logic:

$\neg$

$\wedge$

$\vee$

$\rightarrow$

$\leftrightarrow$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- The main points to remember:
  - ¬ binds to whatever immediately follows it.
  - ∧ and ∨ bind more tightly than →.
  - We will commonly write expressions like $p \wedge q \rightarrow r$ without adding parentheses.
- *For more complex expressions, let's agree to use parentheses!*

The Big Table

Connective	Read Aloud As	C++ Version	Fancy Name
¬	“not”	!	Negation
∧	“and”	&&	Conjunction
∨	“or”	||	Disjunction
→	“implies” or “if…then”	*see PS2!*	Implication
↔	“if and only if”	*see PS2!*	Biconditional
⊤	“true”	`true`	Truth
⊥	“false”	`false`	Falsity

Recap So Far

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are
  - Negation: $\neg p$
  - Conjunction: $p \wedge q$
  - Disjunction: $p \vee q$
  - Implication: $p \rightarrow q$
  - Biconditional: $p \leftrightarrow q$
  - True: $\top$
  - False: $\bot$

Translating into Propositional Logic

Some Sample Propositions

a: I will be in the path of totality.

b: I will see a total solar eclipse.

Some Sample Propositions

a: I will be in the path of totality.

b: I will see a total solar eclipse.

Quick check: How would you write this in propositional logic? "I won't see a total solar eclipse if I'm not in the path of totality."

Go to PollEv.com/cs103spr26

Some Sample Propositions

a: I will be in the path of totality.

b: I will see a total solar eclipse.

"I won't see a total solar eclipse if I'm not in the path of totality."

$$\neg a \rightarrow \neg b$$

“*p* if *q*”

translates to

$$q \rightarrow p$$

It does *not* translate to

 $p \rightarrow q$ 

Some Sample Propositions

a: I will be in the path of totality.

b: I will see a total solar eclipse.

c: There is a total solar eclipse today.

Some Sample Propositions

a: I will be in the path of totality.

b: I will see a total solar eclipse.

c: There is a total solar eclipse today.

“If I will be in the path of totality, but there's no solar eclipse today, I won't see a total solar eclipse.”

Some Sample Propositions

a: I will be in the path of totality.

b: I will see a total solar eclipse.

c: There is a total solar eclipse today.

"If I will be in the path of totality, but there's no solar eclipse today, I won't see a total solar eclipse."

$$(a \wedge \neg c) \rightarrow \neg b$$

“***p*, but *q***”

translates to

$$p \wedge q$$

The Takeaway Point

- When translating into or out of propositional logic, be very careful not to get tripped up by nuances of the English language.
  - In fact, this is one of the reasons we have a symbolic notation in the first place!
- Many prepositional phrases lead to counterintuitive translations; make sure to double-check yourself!

Propositional Equivalences

Quick Question:

What would I have to show you to convince you that the statement $\boldsymbol{p \wedge q}$ is false?

p = “there is chocolate under Cup 1”

q = “there is a chocolate under Cup 2”

Quick Question:

What would I have to show you to convince you that the statement $p \vee q$ is false?

p = "there is chocolate under Cup 1"

q = "there is a chocolate under Cup 2"

Quick check:

(a) Lift Cup 1 and see candy
(b) Lift Cup 2 and see candy
(c) both (a) and (b)
(d) Lift Cup 1 and see empty
(e) Lift Cup 2 and see empty
(f) both (d) and (e)
(g) something else

Go to PollEv.com/cs103spr26

DeMorgan's Laws

- Using truth tables, we concluded that

$$\neg(p \wedge q)$$

  is equivalent to

$$\neg p \vee \neg q$$

- We also saw that

$$\neg(p \vee q)$$

  is equivalent to

$$\neg p \wedge \neg q$$

- These two equivalences are called ***De Morgan's Laws***.

DeMorgan's Laws in Code

- ***Pro tip:*** Don't write this:

```
if (!(p() && q())) {
    /* … */
}
```

- Write this instead:

```
if (!p() || !q()) {
    /* … */
}
```

- (This even short-circuits correctly!)

An Important Equivalence

- Earlier, we talked about the truth table for $p \to q$. We chose it so that

  $$p \to q \quad \textit{is equivalent to} \quad \neg(p \wedge \neg q)$$

- Later on, this equivalence will be incredibly useful:

  $$\neg(p \to q) \quad \textit{is equivalent to} \quad p \wedge \neg q$$

Another Important Equivalence

- Here's a useful equivalence. Start with

$$p \rightarrow q \quad \textit{is equivalent to} \quad \neg(p \wedge \neg q)$$

- By DeMorgan's laws:

$$p \rightarrow q \quad \textit{is equivalent to} \quad \neg(p \wedge \neg q)$$

$$\textit{is equivalent to} \quad \neg p \vee \neg\neg q$$

$$\textit{is equivalent to} \quad \neg p \vee q$$

- Thus $p \rightarrow q$ is equivalent to $\neg p \vee q$

Another Important Equivalence

- Here's a useful equivalence. Start with

  $p \rightarrow q$ *is equivalent to* $\neg(p \wedge \neg q)$

- By de Morgan's laws:

  $p \rightarrow q$ *is equivalent*

  is equivalent

  is equivalent

If p is false, then $\neg p \vee q$ is true. If p is true, then q has to be true for the whole expression to be true.

- Thus $p \rightarrow q$ is equivalent to $\neg p \vee q$

Next Time

- ***First-Order Logic***
  - Reasoning about groups of objects.
- ***First-Order Translations***
  - Expressing yourself in symbolic math!